Neighborhood Properties of Ruscheweyh Type Analytic Functions

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ABSTRACT: In this paper, we define the new subclasses $S_{m,n}(\beta,\gamma,\delta)$, $R_{m,n}(\beta,\gamma,\delta\,;\,\mu)$, $S_{m,n}(\alpha,\beta,\gamma,\delta)$ and $R_{m,n}(\alpha,\beta,\gamma,\delta\,;\,\mu)$ of $T(n)$ using generalized Ruscheweyh derivative and certain properties of neighborhoods for functions belonging to these classes are studied.

Key words and phrase: Univalent functions, Neighborhoods, Convex functions, Starlike functions and Ruscheweyh derivative.

1. INTRODUCTION

Let $T(n)$ denote the family of analytic functions defined in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, (a_k \geq 0, k \in \mathbb{N}\{1\}, n \in \mathbb{N}) \quad (1.1)$$

For any function $f \in T(n)$ and $\lambda \geq 0$,we define

$$N_{n,\lambda}(f) = \{g \in T(n): g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \ and \ \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \lambda\} \quad (1.2)$$

which is the (n,λ) - neighborhood of $f(z)$.

For $e(z) = z$, we see that

$$N_{n,\lambda}(e) = \{g \in T(n): g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \ and \ \sum_{k=n+1}^{\infty} k|b_k| \leq \lambda\}. \quad (1.3)$$

The concept of neighborhoods was first introduced by Goodman [4] and then generalized by Ruscheweyh [6].

Let $S_n^*(\gamma)$ [1] denote the subclass of $T(n)$, defined by the functions of complex order satisfying,

$$\Re\left\{1 + \frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)} - 1\right]\right\} > 0, (z \in U, \gamma \in \mathbb{C}\backslash\{0\}). \quad (1.4)$$

The subclass $C_n(\gamma)$[1] of $T(n)$, is the class of functions of complex order satisfying,

$$\Re\left\{1 + \frac{1}{\gamma}\left[\frac{zf''(z)}{f'(z)}\right]\right\} > 0, (z \in U, \gamma \in \mathbb{C}\backslash\{0\}). \quad (1.5)$$

For $\delta \geq 0$, and $m \in \mathbb{N}_0$, we introduce a linear operator $\mathcal{D}_\delta^m$ defined by

$$\mathcal{D}_\delta^m f(z) = [(k_\delta * k_\delta * \ldots * k_\delta) * f](z), \ z \in \mathcal{U}, \quad (1.6)$$

where $k_\delta = z(1-z)^{-\delta-1}$.

For function $f \in T(n)$ of the form (1.1), we have

$$\mathcal{D}_\delta^m f(z) = z - \sum_{k=n+1}^{\infty} B_\delta(k,m) a_k z^k, \quad (1.7)$$

where

$$B_\delta(k,m) = \left[\frac{\Gamma(k+\delta)}{\Gamma(k)\Gamma(1+\delta)}\right]^m. \quad (1.8)$$

For $m = 1, \ and \ \delta = 1$ the linear operator $\mathcal{D}_\delta^m$ reduces to Ruscheweyh derivative and Sălăgean differential operator respectively.

Now using generalize Ruscheweyh derivative, we define the following subclasses of $T(n)$.

Let $S_{m,n}(\beta,\gamma,\delta)$ denote the subclass of $T(n)$ *defined by* the class of analytic functions f satisfying

$$\left|\frac{1}{\gamma}\left(\frac{z(\mathcal{D}_\delta^m f(z))'}{\mathcal{D}_\delta^m f(z)} - 1\right)\right| < \beta, \quad (1.9)$$

where $m \in \mathbb{N}_0, \gamma \in \mathbb{C}\backslash\{0\}, 0 < \beta \leq 1 \ and \ z \in \mathcal{U}$.

Let $R_{m,n}(\beta,\gamma,\delta\,;\,\mu)$ denote the subclass of $T(n)$ *defined by* the family of analytic functions f such that

$$\left|\frac{1}{\gamma}\left((1-\mu)\frac{\mathcal{D}_\delta^m f(z)}{z} + \mu(\mathcal{D}_\delta^m f(z))' - 1\right)\right| < \beta, \quad (1.10)$$

where $m \in \mathbb{N}_0, \gamma \in \mathbb{C}\backslash\{0\}, 0 < \beta \leq 1 \ and \ z \in \mathcal{U}$.

In the following sections, we derive certain properties of (n,λ)-neighborhoods for analytic functions of complex order in $\mathcal{U}$.

2. MAIN RESULTS

Lemma2.1: A function $f \in S_{m,n}(\beta,\gamma,\delta)$ if and only if

$$\sum_{k=n+1}^{\infty} B_\delta(k,m)[\beta|\gamma|+k-1]a_k < \beta|\gamma|, \qquad (2.1)$$

where $B_\delta(k,m)$ *is given in* (1.8).

Proof: Suppose $f \in S_{m,n}(\beta,\gamma,\delta)$. Then by (1.9) we have

$$\Re\left\{\frac{z\left(\mathcal{D}_\delta^m f(z)\right)'}{\mathcal{D}_\delta^m f(z)}-1\right\} > -\beta|\gamma|, \ (z\in\mathcal{U}). \ (2.2)$$

Using (1.1) and (1.7), we have,

$$\Re\left\{\frac{\sum_{k=n+1}^{\infty} B_\delta(k,m)a_k z^k[1-k]}{z-\sum_{k=n+1}^{\infty} B_\delta(k,m)a_k z^k}\right\} > -\beta|\gamma|.$$

Let $z\to 1$, through the real values in the above inequality we get

$$\sum_{k=n+1}^{\infty} B_\delta(k,m)a_k[1-k] > -\beta|\gamma|\left(1-\sum_{k=n+1}^{\infty} B_\delta(k,m)a_k\right)$$

This Inequality yields the desired condition.

Conversely, by (2.1) and letting |z|=1 we obtain,

$$\left|\frac{z\left(\mathcal{D}_\delta^m f(z)\right)'}{\mathcal{D}_\delta^m f(z)}-1\right| = \left|\frac{\sum_{k=n+1}^{\infty} B_\delta(k,m)a_k z^k[1-k]}{z-\sum_{k=n+1}^{\infty} B_\delta(k,m)a_k z^k}\right|$$

$$\le \frac{\beta|\gamma|\left(1-\sum_{k=n+1}^{\infty} B_\delta(k,m)a_k(k-1)\right)}{1-\sum_{k=n+1}^{\infty} B_\delta(k,m)a_k}$$

$$\le \beta|\gamma|.$$

Hence by the maximum modulus theorem, we have $f \in S_{m,n}(\beta,\gamma,\delta)$, which established the required result.

Analogously, we have the following Lemma.

Lemma2.2: A function $f \in R_{m,n}(\beta,\gamma,\delta\,;\,\mu)$ if and only if

$$\sum_{k=n+1}^{\infty} B_\delta(k,m)[\mu(k-1)+1]a_k < \beta|\gamma|. \qquad (2.3)$$

Theorem2.3: If $\lambda = \frac{\beta|\gamma|(n+1)}{B_\delta(n+1,m)[\beta|\gamma|+n]}$, $(|\gamma|<1)$, then $S_{m,n}(\beta,\gamma,\delta) \subset \mathcal{N}_{n,\lambda}(e)$.

Proof: Let $f \in S_{m,n}(\beta,\gamma,\delta)$. Then by Lemma 2.1, we have

$$\sum_{k=n+1}^{\infty} a_k \le \frac{\beta|\gamma|}{B_\delta(n+1,m)[\beta|\gamma|+n]}. \qquad (2.4)$$

Using (2.1) and (2.4), we have,

$$B_\delta(n+1,m)\sum_{k=n+1}^{\infty} k a_k \le$$

$$\beta|\gamma| + (1-\beta|\gamma|)B_\delta(n+1,m)\sum_{k=n+1}^{\infty} a_k$$

$$\le \frac{\beta|\gamma|(n+1)}{[\beta|\gamma|+n]}.$$

i.e., $\sum_{k=n+1}^{\infty} k a_k \le \frac{\beta|\gamma|(n+1)}{B_\delta(n+1,m)[\beta|\gamma|+n]} = \lambda.$

Thus from (1.3), $f \in \mathcal{N}_{n,\lambda}(e)$.

Hence the proof.

Thereom2.4: If $\lambda = \frac{\beta|\gamma|(n+1)}{B_\delta(n+1,m)[\mu n+1]}$, $(|\gamma|<1)$, then $R_{m,n}(\beta,\gamma,\delta\,;\,\mu) \subset \mathcal{N}_{n,\lambda}(e)$.

Proof: Let $f \in R_{m,n}(\beta,\gamma,\delta\,;\,\mu)$. Then by Lemma 2.2, we have

$$\sum_{k=n+1}^{\infty} a_k \le \frac{\beta|\gamma|}{B_\delta(n+1,m)[\mu n+1]}. \qquad (2.5)$$

From (2.3) and (2.5), we have,

$$\mu B_\delta(n+1,m)\sum_{k=n+1}^{\infty} k a_k \le$$

$$\beta|\gamma| + (\mu-1)B_\delta(n+1,m)\sum_{k=n+1}^{\infty} a_k$$

$$\le \frac{\beta|\gamma|\mu(n+1)}{[\mu n+1]}.$$

That is,

$$\sum_{k=n+1}^{\infty} k a_k \le \frac{\beta|\gamma|(n+1)}{B_\delta(n+1,m)[\mu n+1]} = \lambda.$$

From (1.3), we have, $f \in \mathcal{N}_{n,\lambda}(e)$, which completes the proof.

3. NEIGHBOURHOODS FOR CLASSES $S_{m,n}(\alpha,\beta,\gamma,\delta)$ *and* $R_{m,n}(\alpha,\beta,\gamma,\delta\,;\,\mu)$

In this section, we define the subclasses $S_{m,n}(\alpha,\beta,\gamma,\delta)$ *and* $R_{m,n}(\alpha,\beta,\gamma,\delta\,;\,\mu)$ of

$T(n)$ and neighborhoods of these classes are obtained.

For $0 \leq \alpha < 1$ *and* $z \in \mathcal{U}$, a function $f \in S_{m,n}(\alpha, \beta, \gamma, \delta)$ if there exists a function $g(z) \in S_{m,n}(\beta, \gamma, \delta)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \alpha. \qquad \textbf{(3.1)}$$

For $0 \leq \alpha < 1$ *and* $z \in \mathcal{U}$, a function $f \in R_{m,n}(\alpha, \beta, \gamma, \delta\,;\, \mu)$ if there exists a function $g(z) \in R_{m,n}(\beta, \gamma, \delta\,;\, \mu)$ such that the inequality (3.1) holds true.

THEOREM3.1: If $g(z) \in S_{m,n}(\beta, \gamma, \delta)$ and

$$\alpha = 1 - \frac{\lambda[\beta|\gamma|+n]\boldsymbol{B}_{\boldsymbol{\delta}}(\boldsymbol{n}+\mathbf{1},\boldsymbol{m})}{(n+1)[(\beta|\gamma|+n)\boldsymbol{B}_{\boldsymbol{\delta}}(\boldsymbol{n}+\mathbf{1},\boldsymbol{m})-\beta|\gamma|]}, \qquad (3.2)$$

then $\boldsymbol{\mathcal{N}}_{\boldsymbol{n},\lambda}(\boldsymbol{g}) \subset S_{m,n}(\alpha, \beta, \gamma, \delta)$.

PROOF: Let $f \in \boldsymbol{\mathcal{N}}_{\boldsymbol{n},\lambda}(\boldsymbol{g})$. Then,

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \lambda, \qquad (3.3)$$

which yields the coefficient inequality,

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leq \frac{\lambda}{n+1}, \quad (n \in \mathbb{N}). \qquad (3.4)$$

Since $g(z) \in S_{m,n}(\beta, \gamma, \delta) by$ (2.4), we have,

$$\sum_{k=n+1}^{\infty} b_k \leq \frac{\beta|\gamma|}{\boldsymbol{B}_{\boldsymbol{\delta}}(\boldsymbol{n}+\mathbf{1},\boldsymbol{m})[\boldsymbol{\beta}|\boldsymbol{\gamma}|+\boldsymbol{n}]}. \qquad (3.5)$$

So that,

$$\left|\frac{\boldsymbol{f}(\boldsymbol{z})}{\boldsymbol{g}(\boldsymbol{z})} - \mathbf{1}\right| < \frac{\sum_{k=n+1}^{\infty} |a_k - b_k|}{\mathbf{1} - \sum_{k=n+1}^{\infty} b_k}$$

$$\leq \frac{\lambda[\beta|\gamma|+n]\boldsymbol{B}_{\boldsymbol{\delta}}(\boldsymbol{n}+\mathbf{1},\boldsymbol{m})}{(n+1)[(\beta|\gamma|+n)\boldsymbol{B}_{\boldsymbol{\delta}}(\boldsymbol{n}+\mathbf{1},\boldsymbol{m})-\beta|\gamma|]},$$

$$= 1 - \alpha.$$

Thus by definition, $f \in S_{m,n}(\alpha, \beta, \gamma, \delta)$ for a given α by (3.2), which establishes the desired result.

On similar lines, we can prove the following theorem.

THEOREM3.2: If $g(z) \in R_{m,n}(\beta, \gamma, \delta\,;\, \mu)$ and

$$\alpha = 1 - \frac{\lambda[\boldsymbol{\mu n}+\mathbf{1}]\boldsymbol{B}_{\boldsymbol{\delta}}(\boldsymbol{n}+\mathbf{1},\boldsymbol{m})}{(n+1)[(\boldsymbol{\mu n}+\mathbf{1})\boldsymbol{B}_{\boldsymbol{\delta}}(\boldsymbol{n}+\mathbf{1},\boldsymbol{m})-\beta|\gamma|]},$$

then, $\boldsymbol{\mathcal{N}}_{\boldsymbol{n},\lambda}(\boldsymbol{g}) \subset R_{m,n}(\alpha, \beta, \gamma, \delta\,;\, \mu)$.

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