Even Vertex Graceful Labeling for Two Graphs

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Abstract

In this paper Even vertex graceful labeling for two graphs namely prism (Dn) and Book (Bn) is defined. For Book (Bn) it is given for the case when 'n' is even

Keywords – Prism, Book, Even vertex gracefulness, Labeling of graphs.

I.Introduction

Let $G = (V, E)$ be a simple graph with a finite non empty set V of 'p' vertices together with set E of q unordered pairs of distinct points of V. Each pair $e = (u_1,u_2)$ of points in E is an edge of G.A graph with p vertices and q edges is called a (p,q) graph. A graph is said to be of order p.

Labeling of a graph G is an assignment of integers either to the vertices of G or edges of G or both subject to certain conditions. A lot of research has been done on various types of labeling by applying to various graphs and still there are many open problems in this field. A current survey of various graph labeling problems can be found in Gallian J.[4]

Definitions and notations required for this paper are mentioned over here.

Definition 1.1: A map $f:V \rightarrow \{0,1,2,----q\}$ is called a graceful labeling if f is one – to – one and the edges receive all the labels from 1 to q where the label of an edge is the absolute value of the difference between vertex labels at its ends. A graph having a graceful labeling is called a graceful graph. [3]

Definition 1.2 : A graph is Even vertex graceful if there exists an injective map $f : E \rightarrow \{ 1,2,---- 2q\}$ so that the induced map $f^* : V \rightarrow \{0,2,4 --- 2k–2\}$ defined by
$f^* (x) = \sum f (xy) \pmod{2k}$ where the sum runs over all edges xy through y and $k = \max(p,q)$ gives distinct labels to all vertices in G

Even vertex graceful labeling for some graphs is discussed by A. Solairaju and P. Muruganantham[6].

Definition1.3:Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2. Then Cartesian products of G_1 and G_2 is denoted by G_1 x G_2. To define the product G_1 x G_2, consider any two   points u= (u_1 , u_2) and V= (v_1 , v_2) in $V = V_1$ x V_2 . Then u and v are adjacent in G_1x G_2 Whenever $u_1= v_1$ and u_2 adj v_2 or $u_2= v_2$ and u_1 adj v_1

Definition 1.4: For $n \geq 3$ prism D_n is the Cartesian product C_n x k_2 where C_n is a cycle on n-vertices and k_2 is the complete graph on 2-vertices.

Definition 1.5 : For $n \geq 3$ the Book B_n with n pages is the certesian product S_n X K_2 where S_n is the star with n end – vertices and K_2 is the complete graph with 2–vertices.
The author has used the terminology and notations of Harary [5] .So , for the terms not defined here and notations not explained here refer to Harary [5] The author has also proven "Even vertex gracefulness of book B_n; when n is odd"[1]

II.MAIN RESULTS :

Theorem 2.1 : Prism D_n is even vertex graceful.
Proof –Firstly define numbering of vertices and edges as follows. Let A:u_1, u_2, ------, u_n, u_1 and B: u_{n+1}, u_{n+2}, ..., u_{2n}, u_{n+1} be two vertex disjoint cycles of D_n such that A is an outer cycle and B is an inner cycle
S: $e_1,e_2,----, e_n$ and T : e_{n+1}, ----, e_{2n} denote edges of outer cycle and inner cycle respectively such that e_i is an edge { u_{i-1}, u_i} for $2 \leq i \leq n$ and $n+2 \leq i \leq 2n$
 Left out edges e_1 and d e_{n+1} are {u_n, u_1} and { u_{2n}, u_{n+1}} respectively.

Further U: e_{2n+1}, e_{2n+2}, -------, e_{3n} denote spokes joining two cycles such that e_{2n+i} is an edge { u_i, u_{n+i} } for all i= 1, 2,----n. Figure 1 shows numbering of Prism D_6
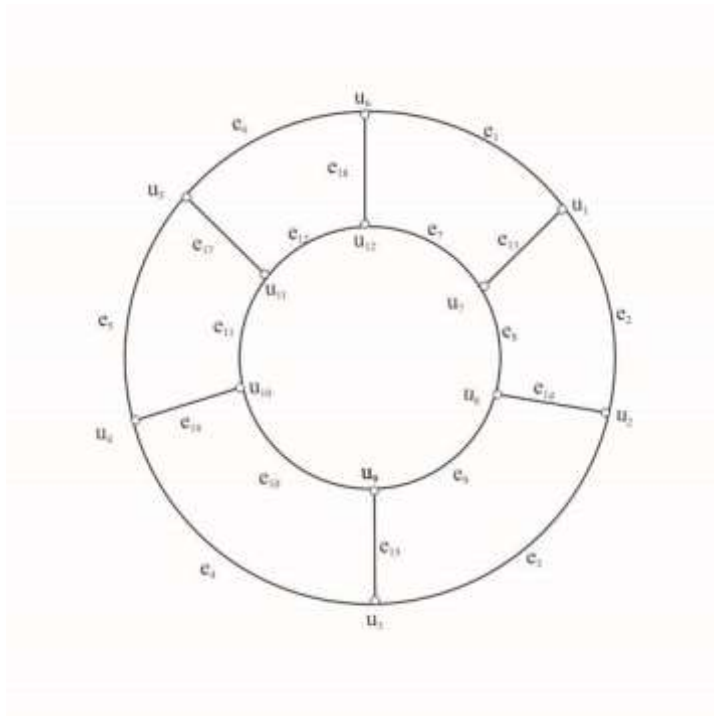

Fig.1 Numbering of D_6

Let f : { e_1, e_2,$e_{........}$,e_{3n}} → {1,2,------, 6n } denote labeling of edges.

Define : - $f(e_i) = 2i – 1$ for $1 \le i \le 2n$

$f(e_{n+i}) = 4n+2-2i$ for $n+1 \le i \le 2n$

Then Edge labels are all distinct and they are

{ 1,3,-----,4n-1} and {2n, 2n-2, 2n-4,-------, 2}

Then Induced Vertex labeling for

u_i, = $f(e_i) + f(e_{i+1}) + f(e_{2n+i})$; $1 \le i \le n-1$

$\equiv (2i-1) + [2(i+1)-1] + 2n +2-2i$ (mod 6n) ; $1 \le i \le n-1$

$\equiv 2n+2i+2$ (mod 6n) ; $1 \le i \le n-1$

$= 2n +2i +2$; $1 \le i \le n-1$

Induced vertex Labels covered are   { $2n + 4$, $2n +6$, $2n+8$,-----, 4n}

Induced vertex labels for $u_n = f(e_1) + f(e_n) + f(e_{3n})$

$\equiv 1 + 2n-1 + 2$ (mod 6n)

$\equiv 2n + 2$ (mod 6n)

$= 2n+2$

Induced Vertex labels for

$u_i = f(e_i)+ f(e_{i+1}) +f(e_{n+i})$; $n+1 \le i \le 2n-1$

$\equiv 2i -1 + 2(i+1) -1 + 4n + 2 – 2i$ (mod 6n) ; $n+1 \le i \le 2n-1$

$\equiv 4n + 2i + 2$ (mod 6n) ; $n+1 \le i \le 2n-1$

Induced Vertex labels covered are 4,6,8,10, -------, 2n (mod6n)

Induced Vertex label for $u_{2n} = f(e_{2n}) + f(e_{3n}) + f(e_{n+1})$

$\equiv 2(2n)-1 + 4n + 2 – 2 (2n) + 2 (n+1) -1$ (mod 6n)

$\equiv 6n+2$ (mod 6n)

$\equiv 2$ (mod 6n)

Hence induced vertex label for $u_{2n} = 2$

Hence Induced vertex label set for D_n is { 2,4,6,------,2n, 2n+2, 2n+4, 2n+6,------, 4n}(mod (6n)) Hence Prism D_n is even vertex graceful.

Remark 2.2 **:** Here important part is the numbering of edges .Due to this the function f becomes simple. If the edges are numbered in a different manner then the labeling function would not be so simple.

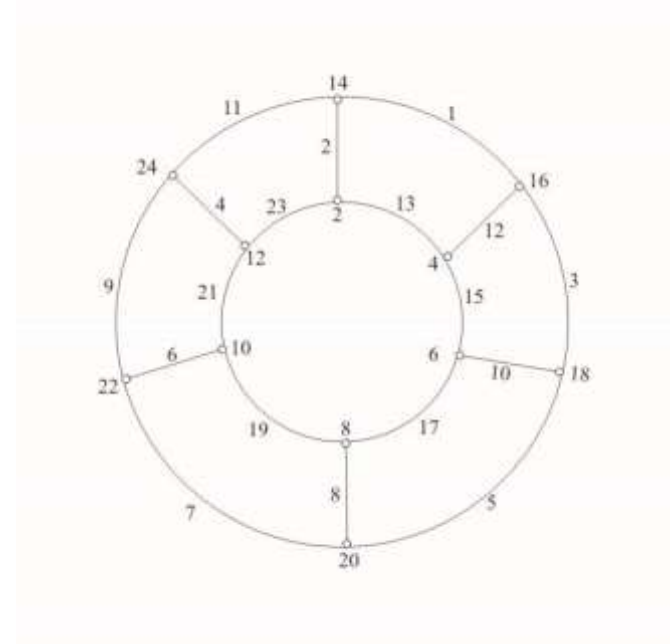*Illustration 2.3***:** Figure 2 shows Even-vertex graceful labeling of Prism D_6

Fig. 2 Even vertex graceful labeling of D_6

Theorem 2.4 : For an even integer $n \ge 2$, book B_n is even vertex graceful

Proof – Here B_n is a book with 'n' number of pages. It is an open book with $n / 2$ pages on left hand side and $n / 2$ pages on right hand side. Each page has 3edges namely upper edge, middle edge and lower edge. Hence book B_n has '3n+ 1' edges .Here also firstly define numbering of edges and vertices. Here numbering of edges is sequential starting from the leftmost page with the upper edge then the middle edge and lastly the lower edge of a page. The middle most edge of the book is numbered as 'e_{3nt1}'

The edges incident at vertex a_i are { e_{3i-2}, e_{3i-1}} and edges incident at vertex bi are

{ e_{3i-1}, e_{3i}} for all i, $1 \le i \le n$

The vertex where all edges e_{3i-2} for $1 \leq i \leq n$ and e_{3nt1} meet is denoted by A. The vertex where all edges e_{3i} for $1 \leq i \leq n$ and e_{3nt1} meet is denoted by B.
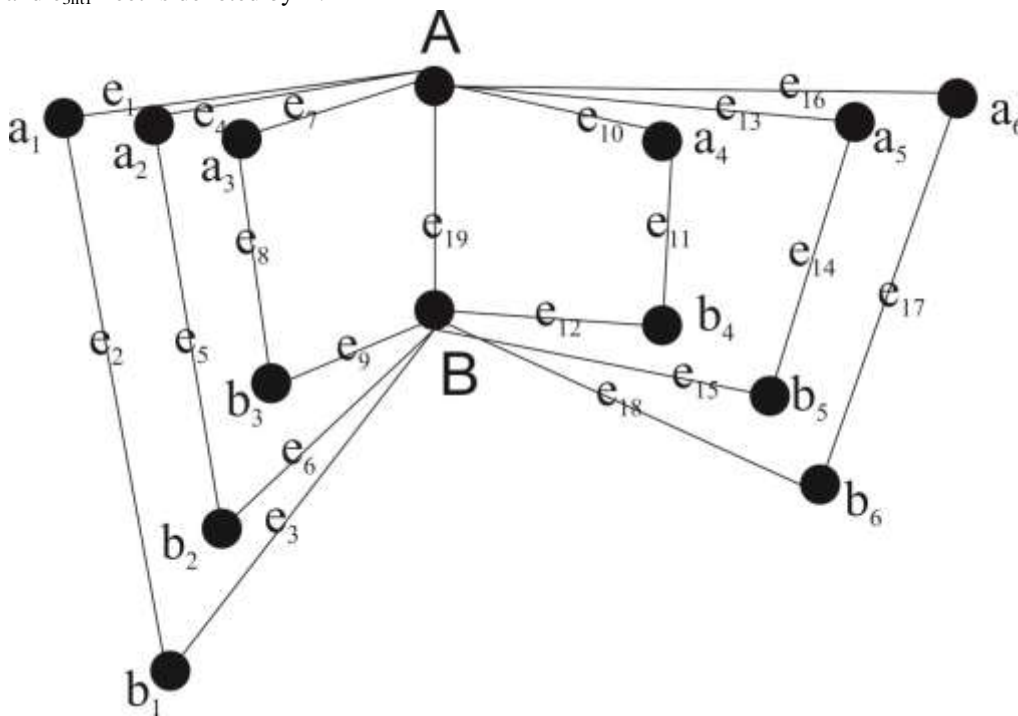
Figure 3 shows numbering of Book B_6

Fig.3 Numbering of B_6

Let the map f : E (B_n) → $\{ 1,2 \ldots , 6n+2\}$ denote labeling of edges and

$f^*(u) = \sum f(uv) \pmod{6n+2}$ where the sum runs over all edges uv through v define induced vertex labeling.

Three different cases are considered; viz n = 0,2,4, (mod 6)

The Case : n ≡ 0 (mod 6)

Define f (e_i) = 2i − 1 for $1 \leq i \leq 3n$ and f (e_{3n+1}) = n − 2

Hence induced vertex labels for a_1, a_2, a_3 …., a_n are 4,16,28,….,12n–8 (mod 6n + 2) respectively.

Similarly induced vertex labels for b_1,b_2…. , b_n are 8,20,32,……,12n – 4 (mod 6n+2) respectively.

Induced vertex label of A

≡1 + 7 + 13 + ……….. + 6n − 5 + n-2 (mod 6n+2)

≡$3n^2 - n - 2$ (mod 6n + 2)

≡ 4n (mod 6n+2)

Hence induced vertex label of A = 4n

Induced vertex label of B

≡5 + 11 + 17 + ……….. + 6n − 1 + n-2 (mod 6n+2)

≡ $3n^2 + 3n - 2$ (mod 6n + 2)

≡ 2n-2 (mod 6n+2)

Hence induced vertex label of B = 2n-2

The Case : n ≡ 2 (mod 6)

Define f (e_1) = 3 , f (e_2) = 1

f(e_i) = 2i − 1 for $3 \leq i \leq 3n - 3$

f (e_{3n-2}) = 6n − 3 , f (e_{3n-1}) = 6n − 5 ,f (e_{3n})= 6n − 1,

 f (e_{3n+1}) = n − 4

 Hence induced vertex labels for a_1, a_2, a_3 …., a_n are 4,16,28,…..,12n–8 (mod 6n + 2) respectively.

Similarly induced vertex labels for b_1,b_2…. , b_n are 8,20,32,……,12n – 4 (mod 6n+2) respectively

Labeling of A and B are as follows

Induced vertex labeling of A

≡ 3 + 7 + 13 + …..+ 6n − 11 + 6n − 3 + n − 4 (mod 6n + 2)

≡$3n^2 - n$ (mod 6n + 2)

 ≡ 4n+2 (mod 6n+2)

Hence induced vertex labeling of A = 4n +2

Next ,induced vertex labeling of B

≡5+11+17+---------+6n-1+n-4 (mod 6n+2)

≡$3n^2$+3n-4(mod 6n+2)

≡ 2n-4 (mod 6n+2)

Hence induced vertex labeling of B = 2n-4

The Case :n≡4 (mod 6)

In this case labeling is similar to that of the case n≡0 (mod 6) except for the middle most edge ,that is, the edge e_{3n+1}.

Therefore induced vertex labeling is also same as that of the case n≡0(mod6) except for the vertices A and B .

Hence in this case only labeling of e_{3n+1} is mentioned and induced vertex labeling of A and B are calculated .

f (e_{3n+1})= n-8

It is obvious that the lable 'n-8' can not be defined for the case n=4.Hence labeling for n=4 ,that is , for Book B_4 is mentioned seperately at the end.

Here we prove it firstly for remaining values , that is, for n ≥ 10

Therefore , induced vertex labeling of A

$\equiv 3n^2 -2n +n- 8 \pmod{6n+2}$

$\equiv 3n^2-n-8 \pmod{6n+2}$

$\equiv 4n-6 \pmod{6n+2}$

Hence induced vertex labeling of A = 4n -6

Next induced vertex ladeling of B $\equiv 3n^2+3n – 8 \pmod{6n+2}$
                    $\equiv 2n-8 \pmod{6n+2}$

 Hence induced vertex label of B = 2n-8

Lastly we give actual labeling of B_4

Figure 4 shows Book B_4 is even vertex graceful

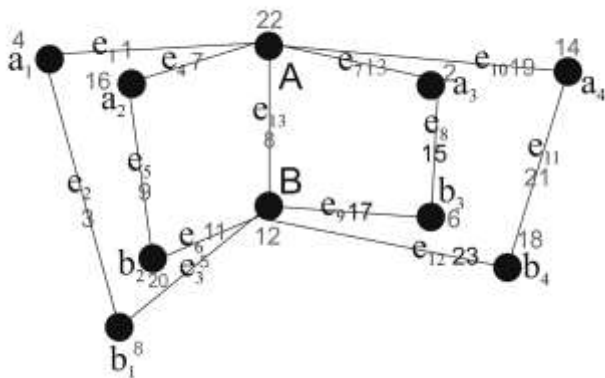Fig. 4 Even vertex graceful labeling of B4

It is clear that here labeling is similar to that of the case n≡0 (mode 6) except for the edge e_{13}

Remark 2.5 **:** Here also numbering of edges and vertices is important. Due to this function f becomes simple. If the edges and the vertices are numbered in different manner then the labeling function would not be so simple.

*ILLUSTRATION 2.6:*Figure 5(on the next page) shows labeling of Book B_8 :The case n ≡ 2(mod 6)

III.Conclusion

 If a graph is symmetric like Prism , Cycle , Corona (Cn * K_1) then most of the labeling functions are simple. Labeling functions also depend upon the way edges and vertices are numbered. The structure of Prism graph suggests that computer programming may be obtained for given labeling. Labeling of discrete structure is a potential area of research due to its diversified applications and it is very interesting to investigate whether any graph or graph family admits a particular labeling or not? Here even vertex graceful labeling for two graphs viz Prism and Book (Bn);n even is obtained.

Open Problems **:** 1) It is possible to obtain 'Even Vertex Graceful Labeling for other graphs

2) There is a scope to find different types of labeling for Prism and Book

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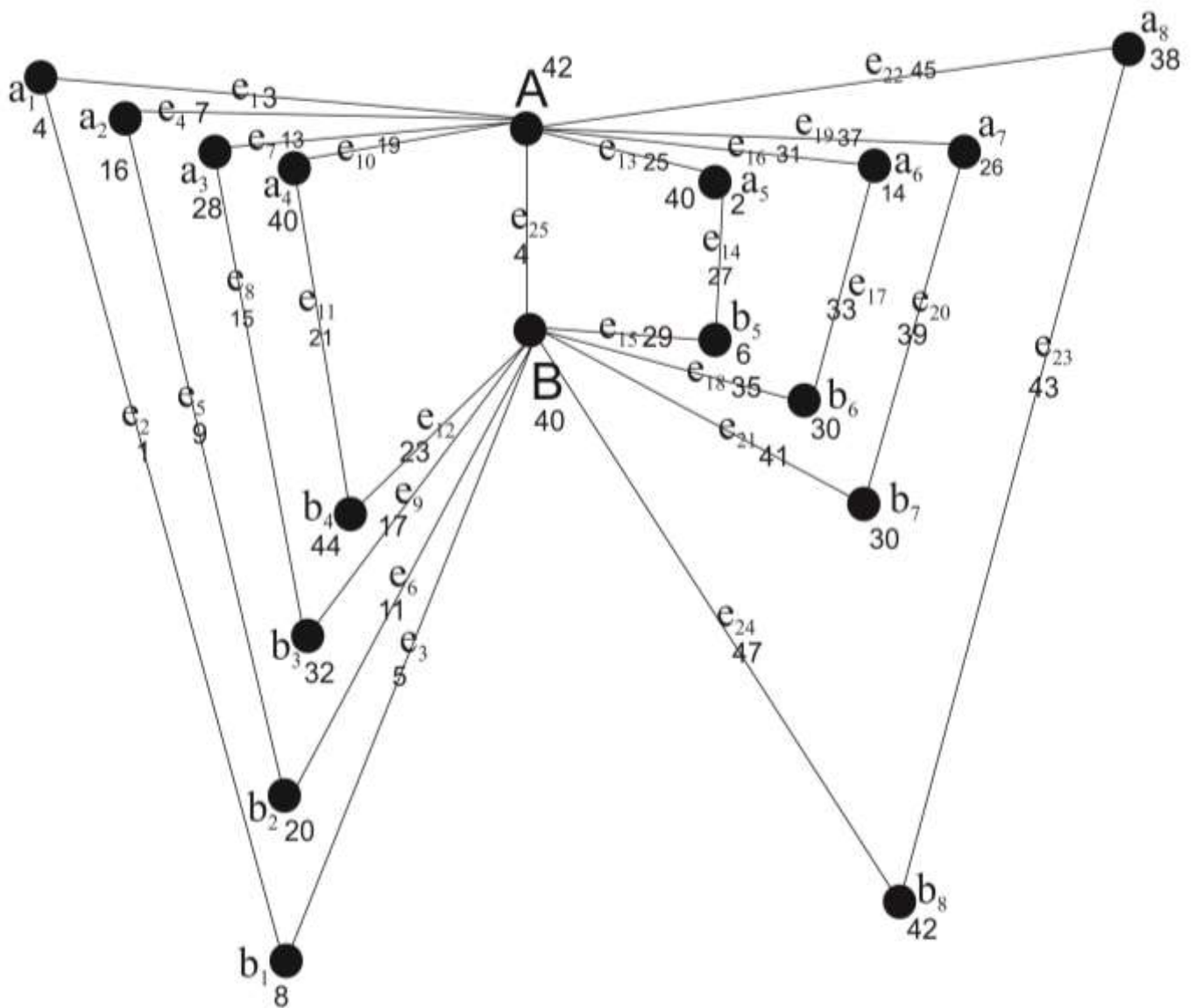
Fig. 5 Even vertex graceful labeling of B₈