Some Graph Labelings on the Inflation of Ladder Graph

K.Thirusangu¹, V.Celin Mary² and D.Suresh³

¹,²Department of Mathematics, SIVET College, Chennai-600073.
³Department of Mathematics, Hindustan College of arts and science, Padur.

Abstract: In this paper, we prove the existence of labelings such as square difference, cube difference square sum and square multiplicative for the inflation of ladder graph.

Keywords: Square difference labeling, Cube difference labeling, Square sum labeling, square multiplicative labeling, inflation graph, ladder graph.

I. Introduction

Let G(V,E) be a simple, finite and undirected graph. Labeling of graphs serve as useful models for broad range of applications such as coding theory, x-ray, crystallography, radar, astronomy, circuit design, communication networks and data base management and models for constraint programming over finite domain. The square sum labeling and square sum graphs are defined and extensively studied by Ajitha, Arumugam and Germina [1]. They proved that the cycle, complete graph Kn, the cycle cactus, ladder and complete lattice grids are square sum graphs. Square sum labeling for some middle and total graphs of some classes of graphs has been studied in the literature [6].

In [7] Shaimadefined new labelings and new graphs such as square difference labelling and square difference graph. He proved that cycles, complete graphs, cycle cactus, ladder, lattice grids, wheels, quadrilateral snakes, the graph G = Kn + mKn are square difference graphs [8]. The concept of cube difference labeling and cube difference graph was introduced in [9]. It is proved that the graphs like paths, cycles, stars, fan graphs, wheel graphs, crown graphs, helm graphs, dragon graphs, coconut trees and shell graphs admit cube difference labelling. Square and Cube Difference Labelling on some special Tree and a New Key Graphs has been studied in the literature [5].

Mirthubashini and Senthil Amuthadefined and studied the concept of square multiplicative labelling [4]. They proved that graphs such as cycle with one chord, cycle with twin chords, quadrilateral triangles, triangular snakes, 2mΔ - snake, double triangular snakes, Bi-star are Square multiplicative graphs. A dynamic survey of graph labeling has been regularly updated by Gallian [3]. In [2] Dunbar and Haynes introduced the concept of the Inflation or Inflated graph G1 of a graph G without isolated vertices is obtained as follows, each vertex xi of degree di of G is replaced by a clique Xi ≅ Kn[di] and each edge xixj of G is replaced by an edge uv in such a way that u ∈ Xi, v ∈ Xj, and two different edges of G are replaced by non adjacent edges of G1.

II. Preliminaries

Definition 2.1: Let G be a (p,q) graph, G is said to be a square difference graph if there exists a bijection f: V(G) → {0,1,...,p-1} such that the induced function f*: E(G) → N given by f*(uv) = |[f(u)]²-[f(v)]²| for every uv ∈ E(G) are all distinct.

Definition 2.2: Let G be a (p,q) graph, G is said to be a cube difference graph if there exists a bijection f: V(G) → {0,1,...,p-1} such that the induced function f*: E(G) → N given by f*(uv) = |[f(u)]³-[f(v)]³| for every uv ∈ E(G) are all distinct.

Definition 2.3: The graph G(V,E) is said to be a square sum graph, if there exist a bijection f: V(G)→{0,1,2,...,p-1} such that the induced function f*: E(G) → N given by f*(uv) = |[f(u)]²+[f(v)]²| for every uv ∈ E(G) are all distinct.

Definition 2.4: The graph G(V,E) is said to be a square multiplicative graph, if there exist a bijection f: V(G)→{1,2,...,p} such that the induced function f*: E(G) → N given by f*(uv) = |[f(u)]²×[f(v)]²| for every uv ∈ E(G) are all distinct.

Definition 2.5: The ladder graph Ln is a planar undirected graph with 2m vertices and 3m – 2 edges. It is obtained as the Cartesian product of two path graphs, one of which has only one edge Lm,1=Pm×P1, where m is the number of rungs in the ladder.
III. Main Results

In this section, we present an algorithm for the structure of inflation of ladder graph and also prove the existence of labeling such as square difference, cube difference, square sum and square multiplicative labeling for the inflated ladder graph.

**Algorithm 3.1:**
Input: Ladder graph

Procedure : Structure of Inflation of ladder graph

\[
V = \{v_1,v_2,\ldots,v_{3n-2},v_1',v_2',\ldots,v_{3n-2}'\} \\
E = \{e_1,e_2,\ldots,e_{2(n-1)},e_1',e_2',\ldots,e_{(4n-5)}\}
\]

for \(i = 1\) to \(n-2\)

\[
v_{i+1}^1 \leftarrow e_{i+(n-1)/2};
\]

end for

\[
v_1 v_{n+1} \leftarrow e_{i+(n-1)/2};
\]

end for

\[
 for \(i = 1\) to \(n\)

\[
v_i v_{i+1} \leftarrow e_{i+(n-1)/2};
\]

end for

\[
 for \(i = 2\) to \(n-1\)

\[
v_i v_{2i+1} \leftarrow e_{i+(3n+2-3)/2};
\]

end for

\[
 v_{n+1} v_{2n+1} \leftarrow e_{i+(3n+2)/2};
\]

end for

\[
 f(v_{i}) = i+1; \quad f(v_{i}^1) = \frac{6n-i}{2} + i; \]

end for

end procedure

Output: Inflation of ladder graph.

**Algorithm 3.2:**
Input: Inflated Ladder Graph \(L_o\)

Procedure (square difference, cube difference and square sum labeling for inflated graph of ladder graph)

\[
V = \{v_{i},v_{i+1},v_{i+2},\ldots,v_{3n-2},v_{i}^1,v_{i+1}^1,v_{i+2}^1,\ldots,v_{3n-2}^1\} \\
E = \{e_1,e_2,\ldots,e_{2(n-1)},e_1',e_2',\ldots,e_{(4n-5)}\}
\]

for \(i = 1\) to \(\frac{6n-4}{2}\)

\[
f(v_i) = i; \quad f(v_i^1) = \frac{6n-i}{2} + i;
\]

end for

end procedure

Output: Labeled Inflated Ladder Graph

**Theorem 3.3:** The inflation of ladder graph \(Inf(L_n)\), is a square difference graph.

**Proof:** From the algorithm 3.1, inflation of Ladder Graph \(L_n\) has \(6n-4\) vertices and \(9n-10\) edges. Let us define a function \(f: V(G) \rightarrow \{0,1,2,\ldots,p-1\}\) by using algorithm 3.2 and get the vertices of \(I(L_n)\) are labeled. In order to get the edge labels, define the induced map \(f': E \rightarrow N\) such that for any \(u,v \in V\), \(f'(u,v) = \|f(u)^2 - f(v)^2\|\). Under this map, the edge labels are as follows:

<table>
<thead>
<tr>
<th>when (n \equiv 0 \pmod{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. for (i = 1) to (\frac{n-2}{2})</td>
</tr>
</tbody>
</table>
|\(f'(v_{i} v_{i+1}) = \|[2i^2 + 2(i-1)^2]|; \quad f'(v_{i+1} v_{i+2}) = \|[3n+2i-4]^2 - [3n+2i-3]^2|\)
|\(f'(v_{n+2} v_{n+1}) = \|[n+2i-1]^2-[n+2i]^2]; \quad f'(v_{n+2} v_{n+3}) = \|\frac{7n+2}{2}^2 - 2i^2 - \left[\frac{7n+2}{2}^2 + 2i+1\right]^2\|\)
|2. for \(i = 2\) to \(n-1\) |
|\(f'(v_{i} v_{i+1}) = \|[i+1]^2 - [2n+i-2]^2]; \quad f'(v_{i+1} v_{2i+1}) = \|[n+i-1]^2 - [2n+i-2]^2|\)
|\(f'(v_{i+1} v_{2i+1}) = \|[3n+i-3]^2 - [5n+i-4]^2]; \quad f'(v_{i} v_{2i+1}) = \|[4n+i-3]^2 - [5n+i-4]^2|\)
|\(f'(v_{i} v_{n+i}) = \|[i+1]^2 - [n+i-1]^2]; \quad f'(v_{n+i} v_{n+i}^1) = \|[3n+i-3]^2 - [4n+i-3]^2|\)
3. for i = 1 to n-2
\[ f^* (v_{2n+i}v_{2n+i}) = [2n+1]^2 - [5n-3]^2 \]
\[ f^* (v_{n+1}v_{n}) = [n-1]^2 \]
\[ f^* (v_{n+1}v_{n}) = [n-1]^2 - [5n-3]^2 \]

when \( n \equiv 1 \pmod{2} \)

1. for i = 1 to \( \frac{n-1}{2} \)
\[ f^* (v_{2n+i}v_{2n+i}) = [2i]^2 - [2i-1]^2 \]
\[ f^* (v_{2n+i}v_{2n+i}) = [3n+1]^2 - [3n+2]^2 \]

2. i = 2 to n-1
\[ f^* (v_{n+1}v_{n}) = [n-1]^2 - [5n-3]^2 \]
\[ f^* (v_{n+1}v_{n}) = [n-1]^2 - [5n-3]^2 \]

From the above two cases, all the edges have distinct values. Hence the Inflation of Ladder \( I_{(n)} \) graph admits Square difference labelling.

**Theorem 3.4:** The inflation of ladder graph \( \text{Inf}(I_{(n)}) \), is a cube difference graph.

**Proof:** From the algorithm 3.1, inflation of Ladder Graph \( I_{(n)} \) has \( 6n-4 \) vertices and \( 9n-10 \) edges. Using algorithm 3.2, define a function \( f : V(G) \to \{0,1,2,\ldots,p\} \) to get labels to the vertices of \( I_{(n)} \). In order to get the edge labels, define the induced map \( f^* : E \to \mathbb{N} \) such that for any \( u,v \in V \), \( f^* (u,v) = [f (u)]^3 - [f (v)]^3 \). Under this map, the edge labels are as follows:

when \( n \equiv 0 \pmod{2} \)

1. for i = 1 to \( \frac{n-2}{2} \)
\[ f^* (v_{2n+i}v_{2n+i}) = [2i]^2 - [2i-1]^2 \]
\[ f^* (v_{n+1}v_{n}) = [n-1]^2 \]
\[ f^* (v_{n+1}v_{n}) = [n-1]^2 - [5n-3]^2 \]

when \( n \equiv 1 \pmod{2} \)

1. for i = 1 to \( \frac{n-1}{2} \)
\[ f^* (v_{2n+i}v_{2n+i}) = [2i]^2 - [2i-1]^2 \]
\[ f^* (v_{n+1}v_{n}) = [n-1]^2 - [5n-3]^2 \]

ISSN: 2231-5373  http://www.ijmttjournal.org  Page 142
Algorithm 3.6:

Hence

From the above two cases, all the edges have distinct values. Hence the Inflation of Ladder \( I(L_n) \) graph admits cube differencelabeling.

Theorem 3.5: The inflation of ladder graph \( In(L_n) \), is a square sum graph.

Proof: From the algorithm 3.1, inflation of Ladder Graph \( I(L_n) \) has \( 6n-4 \) vertices and \( 9n-10 \) edges. By using algorithm 3.2, define a function \( f: V(G)\rightarrow \{0,1,2,\ldots, p \} \) for any \( u,v\in V \), \( f * (u,v) = |f(u)|^2 + |f(v)|^2 \). Under this map, the edge labels are as follows:

when \( n \equiv 0 \, (mod \, 2) \)

1. for \( i=1 \) to \( n-2 \)

2. for \( i=2 \) to \( n-1 \)

3. for \( i=1 \) to \( n-2 \)

4. for \( i=1 \) to \( n-2 \)

From the above two cases, all the edges have distinct values. Hence the Inflation of Ladder \( I(L_n) \) graph admits Square sum labeling.

Algorithm 3.6:
Input: Inflation of Ladder graph \( L_n \)

Procedure: square multiplicative labelling for inflated graph of ladder graph

\[ V \leftarrow \{ v_1, v_2, \ldots, v_{3n-2}, v_1, v_2, \ldots, v_{3n-2} \} \]

\[ E \leftarrow \{ e_1, e_2, \ldots, e_{5n-1}, e_1, e_2, \ldots, e_{4n-5} \} \]

for \( i = 1 \) to \( n \)

\[ f(v_i) \leftarrow i; \]
\[ f(v_{3n-i}) \leftarrow 2n+i; \]
\[ f(v_{4n-i}) \leftarrow 3n+i; \]

end for

for \( i = 1 \) to \( n-2 \)

\[ f(v_{2n+i}) \leftarrow 4n+i; \]
\[ f(v_{2n+i}) \leftarrow 5n-2+i; \]

end for

output: LabeledInflation of Ladder Graph.

**Theorem 3.7:** The inflation of ladder graph \( \text{Inf}(L_n) \), is a square multiplicative graph.

**Proof:** From the algorithm 3.1, inflation of ladder graph \( \text{Inf}(L_n) \) has \( 6n-4 \) vertices and \( 9n-10 \) edges. To label the vertices, define a function \( f : V(G) \rightarrow \{ 1, 2, \ldots, p \} \), using algorithm 3.6. In order to get the edge labels, define the induced map \( f^* : E \rightarrow \mathbb{N} \) such that for any \( u, v \in V \), \( f^* (u, v) = [f(u)]^2 \times [f(v)]^2 \). Under this map, the edge labels are as follows:

**when \( n \equiv 0 \) (mod 2)**

1. for \( i = 1 \) to \( \frac{n}{2} \)

   \[ f^* (v_{2i-1}, v_{2i}) = [2i]^2 \times [2i-1]^2 ; f^* (v_{2i-1}, v_{2i+1}) = [n+2i-1]^2 \times [n+2i]^2 \]

2. for \( i = 1 \) to \( \frac{n-2}{2} \)

   \[ f^* (v_{2n+1}, v_{2n+2}) = [2n+2i]^2 \times [2n+2i+1]^2 ; f^* (v_{2n+1}, v_{2n+3}) = [3n+2i]^2 \times [3n+2i+1]^2 \]

3. for \( i = 1 \) to \( n-2 \)

   \[ f^* (v_{3n+1}, v_{3n+2}) = [n+1]^2 \times [n+2i]^2 ; f^* (v_{3n+1}, v_{3n+3}) = [n+1]^2 \times [3n+1]^2 ; f^* (v_{3n+1}, v_{3n+4}) = [n+2i]^2 \times [3n+2i]^2 ; f^* (v_{3n+1}, v_{3n+5}) = [3n]^2 \times [4n]^2 \]

**when \( n \equiv 1 \) (mod 2)**

1. for \( i = 1 \) to \( \frac{n-1}{2} \)

   \[ f^* (v_{2i-1}, v_{2i}) = [2i]^2 \times [2i-1]^2 ; f^* (v_{2i-1}, v_{2i+1}) = [n+2i-1]^2 \times [n+2i]^2 \]

2. for \( i = 1 \) to \( \frac{n-1}{2} \)

   \[ f^* (v_{2n+1}, v_{2n+2}) = [2n+2i]^2 \times [2n+2i+1]^2 ; f^* (v_{2n+1}, v_{2n+3}) = [3n+2i]^2 \times [3n+2i+1]^2 \]

3. for \( i = 1 \) to \( n-2 \)

   \[ f^* (v_{3n+1}, v_{3n+2}) = [n+1]^2 \times [n+2i]^2 ; f^* (v_{3n+1}, v_{3n+3}) = [n+1]^2 \times [3n+1]^2 ; f^* (v_{3n+1}, v_{3n+4}) = [n+2i]^2 \times [3n+2i]^2 ; f^* (v_{3n+1}, v_{3n+5}) = [3n]^2 \times [4n]^2 \]
\[ f^{*}(v_{n+1}v_{n+1}) = [2n+1]^2 \times [3n+1]^2; f^{*}(v_{n}v_{1}) = [n]^2 \times [2n]^2 \]

From the above two cases, all the edges have distinct values. Hence the Inflation of Ladder \( I(L_n) \) graph admits Square multiplicative labeling.

**Illustration:**
The square difference labeling, cube difference labeling and square sum labelling for the inflation of ladder graph \( \text{Inf}(L_n) \) is shown in the figure 4.1. and the square multiplicative labeling for the inflation of ladder graph \( \text{Inf}(L_n) \) is shown in the figure 4.2.

![Figure 4.1](image1)

![Figure 4.2](image2)

**IV. Conclusion:**
We proved that the inflation of ladder graph is square difference graph, cube difference graph square sum and square multiplicative graph.

**References**