Comparison of Openness In Two Fuzzy Topological Spaces And Its Associated Intuitionistic Fuzzy Topological Space

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Abstract: We have that an IFTS can be associated with two fuzzy topological spaces and vice versa [1]. If \((X, T)\) is an IFTS and \(T_1 = \{ \mu / \exists \gamma \in I^X \text{ such that } (\mu, \gamma) \in T \} \), \(T_2 = \{ 1 - \gamma / \exists \mu \in I^X \text{ such that } (\mu, \gamma) \in T \} \), it is easy to see that \((X, T_1)\) and \((X, T_2)\) are fuzzy topological spaces. Similarly if \((X, T_1)\) and \((X, T_2)\) are two fuzzy topological spaces, \(T = \{(u, v') / u \in T_1, v \in T_2 \text{ and } u \subseteq v \}\) is an IFT and \((X, T)\) is an IFTS. In this paper we analyze whether alpha, beta, semi, pre, semi openness of an intuitionistic fuzzy set \((A_1, A_2)\) in an intuitionistic fuzzy topological space will lead to the corresponding openness of co-ordinate fuzzy sets \(A_1\) and \(A_2\) in the corresponding fuzzy topological spaces.

2. Preliminaries

Definition 2.1: [2]
Let an ordinary non fuzzy set \(X\) be given. An intuitionistic fuzzy set (IFS in short) \(A\) in \(X\) has the form \(A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X \}\), where \(\mu_A : X \rightarrow [0,1]\) and \(\gamma_A : X \rightarrow [0,1]\) are functions defining membership and non membership, respectively of each element in the set \(X\). Moreover for each \(x \in X\), the inequality \(\mu_A(x) + \gamma_A(x) \leq 1\) is fulfilled.

Definition 2.2: [2]
Let \(Y\) be a nonempty set. Let \(A\) and \(B\) be intuitionistic fuzzy sets. Let \(A_\alpha = \{\alpha \in \land \text{ such that } (\mu_A(x), \gamma_A(x)) \in T \}\) be an arbitrary family of intuitionistic fuzzy sets in \(X\). Then
(i) \(A \subseteq B\) if \(\forall y \in Y [\mu_A(y) \leq \mu_B(y) \text{ and } \gamma_A(y) \geq \gamma_B(y)]\).
(ii) \(A = B\) if \(A \subseteq B\) and \(B \subseteq A\).
(iii) \(A^c = \{ y : \gamma_A(y) = 0 \} \).
(iv) \(0 = \{ y : \gamma_A(y) = 0 \} \).
(v) \(1 = \{ y : \gamma_A(y) = 1 \} \).
(vi) \(\cap A_\alpha = \{ y : \cap \mu_A(y) \geq \gamma_A(y) \} \).
(vii) \(\cup A_\alpha = \{ y : \cap \mu_A(y) \leq \gamma_A(y) \} \).

Definition 2.3 [3]
An intuitionistic fuzzy topology on a nonempty set \(X\) is a family \(T\) of intuitionistic fuzzy sets in \(X\) which satisfy the following axioms.
(i) \(0, 1 \in T\)
(ii) \(A_1 \cap A_2 \in T\) for any \(A_1, A_2 \in T\).
(iii) \(A_\alpha\) for any arbitrary family \(A_\alpha : \alpha \in \land \in T\).
In this case the pair $(X, T)$ is called an **intuitionistic fuzzy topological space** (IFTS in short) and any IFS in $T$ is known as **intuitionistic fuzzy open set** in $X$. An intuitionistic fuzzy set $K$ is called **intuitionistic fuzzy closed set in $X$** if $K^C \in T$.

**Definition 1.15** : [3]

Let $(X, T_1)$, $(X, T_2)$ be two IFTSs on $X$. Then $T_1$ is said to be contained in $T_2$ if $G \in T_2$ for each $G \in T_1$. In this case we also say that $T_1$ is coarser than $T_2$ or $T_2$ is finer than $T_1$.

**Definition 2.4** : [3]

Let $(X, T)$ be an IFTS and $A$ be an IFS in $X$. Then the **intuitionistic fuzzy interior** and **intuitionistic fuzzy closure** of $A$ are defined by

\[
\text{Cl}(A) = \cap \{K: K \text{ is an IFCS in } X \text{ and } A \subseteq K\}
\]

\[
\text{Int}(A) = \cup \{G: G \text{ is an IFOS in } X \text{ and } G \subseteq A\}.
\]

**Definition 2.5** : [5]

An IFS $A$ in an IFTS $(X, T)$ is said to be

(i) **intuitionistic fuzzy semi open** if $A \subseteq \text{cl} (\text{Int} A)$

(ii) **intuitionistic fuzzy α-open** if $A \subseteq \text{int} (\text{cl} (\text{Int} A))$

(iii) **intuitionistic pre open** if $A \subseteq \text{int} (\text{cl} A)$

(iv) **intuitionistic fuzzy β-open** if $A \subseteq \text{cl} (\text{Int} (\text{cl} (\text{Int} A)))$.

An intuitionistic fuzzy set $A$ is said to be intuitionistic fuzzy semi closed (respectively intuitionistic fuzzy α-closed, intuitionistic fuzzy pre closed, intuitionistic fuzzy β-closed) if $A^C$ is intuitionistic fuzzy semi open (respectively intuitionistic fuzzy α-open, intuitionistic fuzzy pre open, intuitionistic fuzzy β-open).

**Result 2.1** : [3]

Let $A$ and $B$ be any two intuitionistic fuzzy sets of an intuitionistic fuzzy topological space $(X, T)$. Then

(i) $A$ is an intuitionistic fuzzy closed set in $X$ if $\text{cl} (A) = A$.

(ii) $A$ is an intuitionistic fuzzy open set in $X$ if $\text{int} (A) = A$.

(iii) $\text{cl}(A^C) = (\text{Int}(A))^C$.

(iv) $\text{int} (A^C) = (\text{cl}(A))^C$.

(v) $A \subseteq B \Rightarrow \text{int}(A) \subseteq \text{int}(B)$.

**Definition 2.6** : [4]

Let $(X, T)$ be an intuitionistic fuzzy topological space. Let $(X, T_1)$ and $(X, T_2)$ be the first and second coordinate fuzzy topological spaces. An IF set $(A_1, A_2)$ is said to be an **I-set** if $\text{Int}_{T}(A_1, A_2) = (\text{Int}_{T_1} A_1, \text{Cl}_{T_2} A_2)$.

**Definition 2.7** : [4]

Let $(X, T)$ be an intuitionistic fuzzy topological space. Let $(X, T_1)$ and $(X, T_2)$ be the first and second coordinate fuzzy topological spaces. An IF Set $(A_1, A_2)$ is said to be a **C-set** if $\text{Cl}_{T}(A_1, A_2) = (\text{Cl}_{T_1} A_1, \text{Int}_{T_2} A_2)$.

**Theorem 2.8** : [4] $(A_1, A_2)$ is an I-set if its complement is a C-set.
Theorem 2.9:[4] Let (X, T) be an IFTS. Let $T_1$, $T_2$ be the co-ordinate fuzzy topologies of $T$.

(i). An IFS $(A_1, A_2)$ is an I-set iff $(\text{Int}_{T_1} A_1, \text{Cl}_{T_2} A_2)$ is open under $T$.

(ii). An IFS $(A_1, A_2)$ is a C-set iff $(\text{Cl}_{T_2} A_1, \text{Int}_{T_1} A_2)$ is a closed under $T$.

3. Comparison of openness In two Fuzzy Topological Spaces And Its Associated Intuitionistic Fuzzy Topological Space

**Theorem 3.1**: Let $(X, T_1)$ and $(X, T_2)$ be two FTSs on $X$, where $T_1 = \{ \mu_i / i \in I \}$ and $T_2 = \{ \rho_i / i \in I \}$. Then for every fuzzy set $A$ on $X$, $\text{Int}_{T_1} A \subseteq \text{Int}_{T_2} A$ iff $T_2$ is finer than $T_1$.

**Proof**: Let $T_2$ be finer than $T_1$. Then $T_1 \sqsubset T_2$. For any fuzzy set $A$ on $X$, $\{ \mu_i \in T_1 / \mu_i \subseteq A \} \subseteq \{ \rho_i \in T_2 / \rho_i \subseteq A \}$ which implies $\max \{ \mu_i \in T_1 / \mu_i \subseteq A \} \subseteq \max \{ \rho_i \in T_2 / \rho_i \subseteq A \}$. Hence $\text{Int}_{T_1} A \subseteq \text{Int}_{T_2} A$.

Conversely, assume that for any fuzzy set $A$ on $X$, $\text{Int}_{T_1} A \subseteq \text{Int}_{T_2} A$.

**Claim**: $T_2$ is finer than $T_1$. Suppose not, then there exists $\mu_i \in T_1$ and $\mu_i \notin T_2$.

Then $\text{Int}_{T_1} \mu_i = \mu_i$. But $\text{Int}_{T_2} \mu_i \neq \mu_i$ (since $\mu_i \notin T_2$). However $\text{Int}_{T_2} \mu_i \subseteq \mu_i$ implies that there exists $x \in X$ such that $\text{Int}_{T_2} \mu_i (x) = \mu_i (x) = \text{Int}_{T_1} \mu_i (x)$. Hence $\text{Int}_{T_1} \mu_i \subseteq \text{Int}_{T_2} \mu_i$. i.e. there exist a fuzzy set $\mu_i$ such that $\text{Int}_{T_1} \mu_i \subseteq \text{Int}_{T_2} \mu_i$ which is a contradiction to the assumption. Hence $T_2$ is finer than $T_1$.

**Theorem 3.2**: Let $(X, T_1)$ and $(X, T_2)$ be two FTSs on $X$. Then for every fuzzy set $A$ on $X$, $\text{Cl}_{T_1} A \supseteq \text{Cl}_{T_2} A$ iff $T_2$ is finer than $T_1$.

**Proof**: Assume that for every fuzzy set $A$ on $X$, $\text{Cl}_{T_1} A \supseteq \text{Cl}_{T_2} A$.

**Claim**: $T_2$ is finer than $T_1$.

Let $A$ be any fuzzy set on $X$. Consider the fuzzy set $A^c$.

By the assumption, $\text{Cl}_{T_1} A^c \supseteq \text{Cl}_{T_2} A^c$.

$\Rightarrow (\text{Int}_{T_1} A)^c \supseteq (\text{Int}_{T_2} A)^c$. (By result 2.1, (iii))

$\Rightarrow \text{Int}_{T_1} A \subseteq \text{Int}_{T_2} A$ for any fuzzy set on $X$. Then by the previous theorem, $T_2$ is finer than $T_1$.

Conversely assume that $T_2$ is finer than $T_1$. Let $A$ be any fuzzy set on $X$.

By the previous theorem, $\text{Int}_{T_1} A^c \subseteq \text{Int}_{T_2} A^c$.

$\Rightarrow (\text{Cl}_{T_1} A)^c \subseteq (\text{Cl}_{T_2} A)^c$. (By result 2.1, (iv)).

$\Rightarrow \text{Cl}_{T_1} A \supseteq \text{Cl}_{T_2} A$ for any fuzzy set on $X$.

**Theorem 3.3**: Let $X$ be a non empty set. Let $(X, T_1)$ and $(X, T_2)$ be two FTSs defined on $X$. Let $(X, T)$ be the associated IFTS. Then the following statements are equivalent.
(i) $T_2$ is finer than $T_1$.

(ii) Every IFS on $X$ is an I-set.

(iii) Every IFS on $X$ is a C-set.

**Proof:**

(i) $\implies$ (ii)

Let $(A_1, A_2)$ be an IFS on $X$.

$T_2$ is finer than $T_1$ $\iff$ $\text{Int}_{T_1} A_1 \subseteq \text{Int}_{T_2} A_1$ (by theorem 3.1).

$\implies$ $\text{Int}_{T_1} A_1 \subseteq \text{Int}_{T_2} (1-A)$ (since $A_1 \subseteq 1-A$).

$\implies$ $\text{Int}_{T_1} A_1 \subseteq (\text{Cl}_{T_2} A_2)^C$. (By result 2.1, (iv))

$\implies$ $(\text{Int}_{T_1} A_1, \text{Cl}_{T_2} A_2)$ is open in $(X, T)$.

$\implies$ $(\text{Int}_{T_1} A_1, \text{Cl}_{T_2} A_2)$ is an I-set.

(ii) $\implies$ (iii)

Assume that every IFS on $X$ is an I-set. Let $(A_1, A_2)$ be an IFS on $X$. By assumption $(A_2, A_1)$ is an I-set. Hence by theorem (2.8), $(A_1, A_2)$ is a C-set.

(iii) $\implies$ (i)

Assume that every IFS on $X$ is a C-set.

We claim that $T_2$ is finer than $T_1$. Suppose not, then by theorem 3.2 there exists a fuzzy set $A$ on $X$ such that $\text{Cl}_{T_2} A \not\subseteq \text{Cl}_{T_1} A$. $\implies$ $\text{Cl}_{T_2} A \not\subseteq (\text{Int}_{T_1} 1-A)^C$. (By result 2.1,(iii))

$\implies$ $\text{Int}_{T_1} 1-A \not\subseteq (\text{Cl}_{T_2} A)^C \implies (\text{Int}_{T_1} 1-A, \text{Cl}_{T_2} A)$ is not open $\implies$ $(\text{Cl}_{T_2} A, \text{Int}_{T_1} 1-A)$ is not closed. Hence $\text{Cl}_T (A, 1-A) \neq (\text{Cl}_{T_2} A, \text{Int}_{T_1} 1-A)$ which implies $(A, 1-A)$ is not a C-set which is a contradiction to the assumption. Hence the theorem.

**Theorem 3.4:** Let $(X, T_1)$ and $(X, T_2)$ be two FTS defined on $X$, where $T_2$ is finer than $T_1$. Let $(X, T)$ be the associated IFTS on $X$. Let $(A_1, A_2)$ be any IFS on $X$. Then the following hold:

(i) $\text{Int}_T (A_1, A_2) = (\text{Int}_{T_1} A_1, \text{Cl}_{T_2} A_2)$.

(ii) $\text{Cl}_T \text{Int}_T (A_1, A_2) = (\text{Cl}_{T_2} \text{Int}_{T_1} A_1, \text{Int}_{T_1} \text{Cl}_{T_2} A_2)$.

(iii) $\text{Int}_T \text{Cl}_T \text{Int}_T (A_1, A_2) = (\text{Int}_{T_1} \text{Cl}_{T_2} \text{Int}_{T_1} A_1, \text{Cl}_{T_2} \text{Int}_{T_1} \text{Cl}_{T_2} A_2)$.

**Proof:**

(i) Holds by the previous theorem.

(ii) To prove, $\text{Cl}_T \text{Int}_T (A_1, A_2) = (\text{Cl}_{T_2} \text{Int}_{T_1} A_1, \text{Int}_{T_1} \text{Cl}_{T_2} A_2)$ it is enough to prove that $(\text{Cl}_{T_2} \text{Int}_{T_1} A_1, \text{Int}_{T_1} \text{Cl}_{T_2} A_2)$ is closed. (By theorem 2.9)
i.e To prove that \((\text{Int}_{T_1} \text{Cl}_{T_2} A_2, \text{Cl}_{T_2} \text{Int}_{T_1} A_1)\) is open.

i.e To prove \(\text{Int}_{T_1} \text{Cl}_{T_2} A_2 \subseteq (\text{Cl}_{T_2} \text{Int}_{T_1} A_1)^C\).

i.e To prove \(\text{Int}_{T_1} \text{Cl}_{T_2} A_2 \subseteq \text{Int}_{T_2}(\text{Int}_{T_1} A_1)^C\). (By result 2.1, (iv))

i.e To prove \(\text{Int}_{T_1} \text{Cl}_{T_2} A_2 \subseteq \text{Int}_{T_2}\text{Cl}_{T_1} A_1^C\). (By result 2.1, (iii))

Now \(T_2\) is finer than \(T_1\). Hence by theorem (3.2), we have \(\text{Cl}_{T_2} A_2 \subseteq \text{Cl}_{T_1} A_2 \subseteq \text{Cl}_{T_1} A_1^C\). (Since \((A_1, A_2)\) is an IFS, \(A_2 \leq A_1^C\)).

Also by theorem (3.1), we have \(\text{Int}_{T_1} \text{Cl}_{T_2} A_2 \subseteq \text{Int}_{T_2} \text{Cl}_{T_2} A_2 \subseteq \text{Int}_{T_2} \text{Cl}_{T_1} A_1^C\). (By result 2.1, (v))

which is to be proved.

Hence \(\text{Cl}_{T_1} \text{Int}_{T_1}(A_1, A_2) = (\text{Cl}_{T_2} \text{Int}_{T_1} A_1, \text{Int}_{T_1} \text{Cl}_{T_2} A_2)\).

(iii) It is enough to prove that \(\text{Int}_{T_1} \text{Cl}_{T_2} \text{Int}_{T_1} A_1 \subseteq (\text{Cl}_{T_2} \text{Int}_{T_1} \text{Cl}_{T_2} A_2)^c\)

i.e To prove \(\text{Int}_{T_1} \text{Cl}_{T_2} \text{Int}_{T_1} A_1 \subseteq \text{Int}_{T_2} (\text{Int}_{T_1} \text{Cl}_{T_2} A_2)^C\). (By result 2.1, (iv))

i.e To prove \(\text{Int}_{T_1} \text{Cl}_{T_2} \text{Int}_{T_1} A_1 \subseteq \text{Int}_{T_2} \text{Cl}_{T_1} (\text{Cl}_{T_2} A_2)^C\). (By result 2.1, (iii))

i.e To prove \(\text{Int}_{T_1} \text{Cl}_{T_2} \text{Int}_{T_1} A_1 \subseteq \text{Int}_{T_2} \text{Cl}_{T_1} \text{Int}_{T_2} A_2^C\). (By result 2.1, (iv))

Since \(T_2\) is finer than \(T_1\), by theorem 3.1, \(\text{Int}_{T_1} A_1 \subseteq \text{Int}_{T_2} A_1 \subseteq \text{Int}_{T_2} A_2^C\). (since \(A_1 \leq A_2^C\)).

Also by theorem 3.2, \(\text{Cl}_{T_2} \text{Int}_{T_1} A_1 \subseteq \text{Cl}_{T_1} \text{Int}_{T_1} A_1 \subseteq \text{Cl}_{T_1} \text{Int}_{T_2} A_2^C\). (By result 1.1, (vi))

Again by theorem 3.1, \(\text{Int}_{T_1} \text{Cl}_{T_2} \text{Int}_{T_1} A_1 \subseteq \text{Int}_{T_2} \text{Cl}_{T_2} \text{Int}_{T_1} A_1 \subseteq \text{Int}_{T_2} \text{Cl}_{T_1} \text{Int}_{T_2} A_2^C\).

(By result 2.1, (v)) which is to be proved.

Hence \(\text{Int}_{T_1} \text{Cl}_{T_1} \text{Int}_{T_1}(A_1, A_2) = (\text{Int}_{T_1} \text{Cl}_{T_2} \text{Int}_{T_1} A_1, \text{Cl}_{T_2} \text{Int}_{T_1} \text{Cl}_{T_2} A_2)\).

**Theorem 3.18:** Let \((X, T_1)\) and \((X, T_2)\) be two FTSs where \(T_2\) is finer than \(T_1\). Let \((X, T)\) be the associated IFTS. Let \((A_1, A_2)\) be an IFS on \(X\). Then

(i) \((A_1, A_2)\) is semi open in \((X, T)\) \(\Rightarrow\) \(A_1\) is semi open in \((X, T_1)\) and in \((X, T_2)\).

(ii) \((A_1, A_2)\) is semi closed in \((X, T)\) \(\Rightarrow\) \(A_2\) is semi open in \((X, T_1)\) and in \((X, T_2)\).

(iii) \((A_1, A_2)\) is pre open in \((X, T)\) \(\Rightarrow\) \(A_1\) is pre open in \((X, T_1)\) and in \((X, T_2)\).

(iv) \((A_1, A_2)\) is pre closed in \((X, T)\) \(\Rightarrow\) \(A_2\) is pre open in \((X, T_1)\) and in \((X, T_2)\).

(v) \((A_1, A_2)\) is \(\alpha\) - open in \((X, T)\) \(\Rightarrow\) \(A_1\) is \(\alpha\) - open in \((X, T_1)\) and in \((X, T_2)\).

(vi) \((A_1, A_2)\) is \(\alpha\) - closed in \((X, T)\) \(\Rightarrow\) \(A_2\) is \(\alpha\) - open in \((X, T_1)\) and in \((X, T_2)\).

(vii) \((A_1, A_2)\) is \(\beta\) - open in \((X, T)\) \(\Rightarrow\) \(A_1\) is \(\beta\) - open in \((X, T_1)\) and in \((X, T_2)\).
(viii) \((A_1, A_2)\) is \(\beta\)-closed in \((X, T)\) \(\Rightarrow A_2\) is \(\beta\) - open in \((X, T_1)\) and in \((X, T_2)\).

**PROOF:**

(i) \((A_1, A_2)\) is semi open in \((X, T)\) \(\Rightarrow (A_1, A_2) \subseteq \text{Cl}_T \text{Int}_T (A_1, A_2)\)

\(\Rightarrow (A_1, A_2) \subseteq (\text{Cl}_{T_1} \text{Int}_{T_1} A_1, \text{Int}_{T_1} \text{Cl}_{T_1} A_2)\) (by (ii) of theorem 3.4) \(\Rightarrow A_1 \subseteq \text{Cl}_{T_2} \text{Int}_{T_2} A_1.\)

\(T_2\) is finer than \(T_1\) \(\Rightarrow \text{Int}_{T_1} A_1 \subseteq \text{Int}_{T_2} A_1.\) (By theorem 3.1).

\(\Rightarrow \text{Cl}_{T_2} \text{Int}_{T_1} A_1 \subseteq \text{Cl}_{T_2} \text{Int}_{T_2} A_1.\) (By result 2.1, (vi))

Also \(T_2\) is finer than \(T_1\) \(\Rightarrow \text{Cl}_{T_2} \text{Int}_{T_1} A_1 \subseteq \text{Cl}_{T_1} \text{Int}_{T_1} A_1.\) (By theorem 3.2).

Hence \(A_1 \subseteq \text{Cl}_{T_1} \text{Int}_{T_1} A_1\) and \(A_1 \subseteq \text{Cl}_{T_2} \text{Int}_{T_2} A_1.\)

\(\Rightarrow A_1\) is semi open in \((X, T_1)\) and in \((X, T_2)\).

(ii) \((A_1, A_2)\) is semi closed in \((X, T)\) \(\Rightarrow (A_2, A_1)\) is semi open in \((X, T)\)

\(\Rightarrow A_2\) is semi open in \((X, T_1)\) and in \((X, T_2)\). (By (i))

(iii) \((A_1, A_2)\) is pre open in \((X, T)\) \(\Rightarrow (A_1, A_2) \subseteq \text{Int}_T \text{Cl}_T (A_1, A_2)\).

\(\Rightarrow (A_1, A_2)^C \supseteq (\text{Int}_T \text{Cl}_T (A_1, A_2))^C.\)

\(\Rightarrow (A_2, A_1) \supseteq \text{Cl}_T (\text{Cl}_T (A_1, A_2))^C.\) (By result 2.1, (iii))

\(\Rightarrow (A_2, A_1) \supseteq \text{Cl}_T \text{Int}_T (A_2, A_1).\) (By result 2.1, (iv))

\(\Rightarrow (A_2, A_1) \supseteq (\text{Cl}_{T_1} \text{Int}_{T_1} A_2, \text{Int}_{T_1} \text{Cl}_{T_1} A_1)\). (By (ii) of theorem 3.4)

\(\Rightarrow A_1 \subseteq \text{Int}_{T_1} \text{Cl}_{T_1} A_1.\)

\(T_2\) is finer than \(T_1\) \(\Rightarrow \text{Cl}_{T_2} A_1 \subseteq \text{Cl}_{T_1} A_1.\) (By theorem 3.2)

\(\Rightarrow \text{Int}_{T_1} \text{Cl}_{T_2} A_1 \subseteq \text{Int}_{T_1} \text{Cl}_{T_1} A_1.\) (By result 2.1, (v)).

Also \(T_2\) is finer than \(T_1\) \(\Rightarrow \text{Int}_{T_1} \text{Cl}_{T_2} A_1 \subseteq \text{Int}_{T_2} \text{Cl}_{T_2} A_1.\)

Hence \(A_1 \subseteq \text{Int}_{T_1} \text{Cl}_{T_1} A_1\) and \(A_1 \subseteq \text{Int}_{T_2} \text{Cl}_{T_2} A_1\)

\(\Rightarrow A_1\) is pre open in \((X, T_1)\) and in \((X, T_2)\).

(iv) \((A_1, A_2)\) is pre closed in \((X, T)\) \(\Rightarrow (A_2, A_1)\) is pre open in \((X, T)\)

\(\Rightarrow A_2\) is pre open in \((X, T_1)\) and in \((X, T_2)\). (By (iii))

(v) \((A_1, A_2)\) is \(\alpha\) – open in \((X, T)\)
\[ (A_1, A_2) \subseteq \text{Int}_T \text{Cl}_T (A_1, A_2) \]

\[ (A_1, A_2) \subseteq (\text{Int}_T \text{Cl}_{T_2} \text{Int}_{T_1} A_1, \text{Cl}_{T_2} \text{Int}_{T_1} \text{Cl}_{T_2} A_2). \]  (By (iii) of theorem 3.4)

\[ A_1 \subseteq \text{Int}_{T_1} \text{Cl}_{T_2} \text{Int}_{T_1} A_1. \]

\( T_2 \) is finer than \( T_1 \) \( \Rightarrow \) \( \text{Cl}_{T_2} \text{Int}_{T_1} A_1 \subseteq \text{Cl}_{T_1} \text{Int}_{T_1} A_1. \)  (By theorem 3.2).

\[ \text{Int}_{T_1} \text{Cl}_{T_2} \text{Int}_{T_1} A_1 \subseteq \text{Int}_{T_1} \text{Cl}_{T_1} \text{Int}_{T_1} A_1. \]  (By result 2.1, (v)).

Also \( T_2 \) is finer than \( T_1 \) \( \Rightarrow \) \( \text{Int}_{T_1} A_1 \subseteq \text{Int}_{T_2} A_1. \)  (By theorem 3.1).

\[ \text{Cl}_{T_2} \text{Int}_{T_1} A_1 \subseteq \text{Cl}_{T_2} \text{Int}_{T_1} A_1. \]  (By result 2.1, (vi)).

\[ \text{Int}_{T_2} \text{Cl}_{T_2} \text{Int}_{T_1} A_1 \subseteq \text{Int}_{T_2} \text{Cl}_{T_2} \text{Int}_{T_2} A_1. \]  (By result 2.1, (v)).

Since \( T_2 \) is finer than \( T_1 \), \( \text{Int}_{T_1} \text{Cl}_{T_2} \text{Int}_{T_1} A_1 \subseteq \text{Int}_{T_2} \text{Cl}_{T_2} \text{Int}_{T_1} A_1 \)

\[ \subseteq \text{Int}_{T_2} \text{Cl}_{T_2} \text{Int}_{T_2} A_1 \]

Hence \( A_1 \subseteq \text{Int}_{T_1} \text{Cl}_{T_1} \text{Int}_{T_1} A_1 \) and \( A_1 \subseteq \text{Int}_{T_2} \text{Cl}_{T_2} \text{Int}_{T_2} A_1 \)

\[ A_1 \text{ is } \alpha- \text{open in } (X, T_1) \text{ and in } (X, T_2). \]

(vi) \( (A_1, A_2) \) is \( \alpha \)-closed in \( (X, T) \) \( \Rightarrow \) \( (A_2, A_1) \) is \( \alpha- \text{open in } (X, T) \)

\[ A_2 \text{ is } \alpha- \text{open in } (X, T_1) \text{ and in } (X, T_2). \]  (By (v))

(vii) \( (A_1, A_2) \) is \( \beta- \text{open in } (X, T) \)

\[ (A_1, A_2) \subseteq \text{Cl}_T \text{Int}_T \text{Cl}_T (A_1, A_2). \]

\[ (A_1, A_2) \subseteq (\text{Cl}_T \text{Int}_T \text{Cl}_T (A_1, A_2))^C. \]

\[ (A_2, A_1) \subseteq \text{Int}_T (\text{Int}_T \text{Cl}_T (A_1, A_2))^C. \]  (By result 2.1, (iv))

\[ (A_2, A_1) \subseteq \text{Int}_T \text{Cl}_T (\text{Cl}_T (A_1, A_2))^C. \]  (By result 2.1, (iii))

\[ (A_2, A_1) \subseteq \text{Int}_T \text{Cl}_T \text{Int}_T (A_1, A_2). \]  (By result 2.1, (iv))

\[ (A_2, A_1) \subseteq (\text{Int}_T \text{Cl}_{T_2} \text{Int}_{T_1} A_2, \text{Cl}_{T_2} \text{Int}_{T_1} \text{Cl}_{T_2} A_1). \]

(By (iii) of theorem 3.4)

\[ A_1 \subseteq \text{Cl}_{T_2} \text{Int}_{T_1} \text{Cl}_{T_2} A_1. \]

\( T_2 \) is finer than \( T_1 \) \( \Rightarrow \) \( \text{Cl}_{T_2} A_1 \subseteq \text{Cl}_{T_1} A_1. \)  (By theorem 3.2)

\[ \text{Int}_{T_1} \text{Cl}_{T_2} A_1 \subseteq \text{Int}_{T_1} \text{Cl}_{T_1} A_1. \]  (By result 2.1, (v)).
\[ \Rightarrow \text{Cl}_{T_2} \text{Int}_{T_1} \text{Cl}_{T_2} A_1 \subseteq \text{Cl}_{T_2} \text{Int}_{T_1} \text{Cl}_{T_1} A_1 \subseteq \text{Cl}_{T_1} \text{Int}_{T_1} \text{Cl}_{T_1} A_1. \]

(By result 2.1, (v) and theorem 3.2).

Also \( T_2 \) is finer than \( T_1 \) \( \Rightarrow \) \( \text{Int}_{T_1} \text{Cl}_{T_2} A_1 \subseteq \text{Int}_{T_2} \text{Cl}_{T_2} A_1. \)

\[ \Rightarrow \text{Cl}_{T_2} \text{Int}_{T_1} \text{Cl}_{T_2} A_1 \subseteq \text{Cl}_{T_2} \text{Int}_{T_2} \text{Cl}_{T_2} A_1. \] (By result 2.1, (v)).

Hence \( A_1 \subseteq \text{Cl}_{T_1} \text{Int}_{T_1} \text{Cl}_{T_1} A_1 \) and \( A_1 \subseteq \text{Cl}_{T_2} \text{Int}_{T_2} \text{Cl}_{T_2} A_1. \)

\[ \Rightarrow A_2 \text{ is } \beta \text{- open in } (X, T_1) \text{ and in } (X, T_2). \]

(viii) \( (A_1, A_2) \) is \( \beta \) closed in \( (X, T) \) \( \Rightarrow \) \( (A_2, A_1) \) is \( \beta \) open in \( (X, T) \)

\[ \Rightarrow A_2 \text{ is } \beta \text{- open in } (X, T_1) \text{ and in } (X, T_2). \] (by (vii)).

References


